

A note on linearized “New Massive Gravity” in arbitrary dimensions

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Abstract

By means of a triple master action we deduce here a linearized version of the “New Massive Gravity” (NMG) in arbitrary dimensions. The theory contains a 4th-order and a 2nd-order term in derivatives. The 4th-order term is invariant under a generalized Weyl symmetry. The action is formulated in terms of a traceless $\eta^{\mu\nu}\Omega_{\mu\nu\rho} = 0$ mixed symmetry tensor $\Omega_{\mu\nu\rho} = -\Omega_{\mu\rho\nu}$ and corresponds to the massive Fierz-Pauli action with the replacement $e_{\mu\nu} = \partial^\rho\Omega_{\mu\nu\rho}$. The linearized 3D and 4D NMG theories are recovered via the invertible maps $\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho}{}^\beta h_{\beta\mu}$ and $\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho}{}^{\gamma\delta} T_{[\gamma\delta]\mu}$ respectively. The properties $h_{\mu\nu} = h_{\nu\mu}$ and $T_{[[\gamma\delta]\mu]} = 0$ follow from the traceless restriction. The equations of motion of the linearized NMG theory can be written as zero “curvature” conditions $\partial_\nu T_{\rho\mu} - \partial_\rho T_{\nu\mu} = 0$ in arbitrary dimensions.

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1 Introduction

An important feature of general relativity is the infinite range of the gravitational interaction mediated by a massless spin-2 particle. It is natural to speculate [1, 2, 3, 4] whether the graviton might have a tiny mass which would certainly have important consequences in the large scale physics of the universe, see [5] for a review work on massive gravity.

The use of the traditional Fierz-Pauli [6] theory for a massive spin-2 particle leads [1, 2] to a conflict, even for a tiny mass, with experimental data for the deviations of light beams by the sun which can be apparently solved by nonlinear self-interaction terms [4]. The required nonlinearities lead on their turn, in general, to the Boulware-Deser ghosts (lack of unitarity). However, recent works, see [7, 8] and also [9], indicate that is possible to cope with unitarity by fine tuning the nonlinear terms.

Another important issue in gravitational interactions is the lack of renormalizability [10, 11]. The addition of higher-derivative terms improve the UV behavior of the graviton propagator but they bring up again the issue of ghosts [12].

From the perspective of both IR and UV modifications of gravity mentioned above, the 3D “New Massive Gravity” (NMG) model [13] plays an interesting role since it contains a massive graviton and higher derivative (4th-order) term simultaneously while keeping the theory unitary even beyond tree level [14]. Moreover, the theory is invariant under general coordinate transformations. All those nice features and its relationship with AdS_3/CFT_2 duality [15] have led to several interesting works.

A step toward a generalization of [13] to $D > 3$ has been taken in [16] where a linearized version of a possible NMG in $D = 4$ is suggested. The model has been shown to be unitary. While the 3D NMG model is usually formulated in terms of a symmetric rank-2 tensor, the 4D model of [16] is given in terms of a rank-3 tensor satisfying $T_{[\mu\nu]\rho} = -T_{[\nu\mu]\rho}$ and $T_{[[\mu\nu]\rho]} = 0$.

Here we generalize the linearized version of the 3D [13] and 4D [16] models to arbitrary dimensions by using a rank-3 tensor which satisfies $\Omega_{\mu\nu\rho} = -\Omega_{\mu\rho\nu}$ and is traceless $\eta^{\mu\nu}\Omega_{\mu\nu\rho} = 0$.

As an introduction to section IV, in sections II and III we derive 4th-order (in derivatives) higher-rank unitary descriptions of spin-0 and spin-1 massive particles via the replacement $\phi \sim \partial_\mu B^\mu$ in the Klein-Gordon action and $A_\mu \sim \partial^\nu W_{\mu\nu}$, with $W_{\mu\nu} = W_{\nu\mu}$, in the Maxwell-Proca theory. The use of a triple master action with sources naturally explain why those simple change of variables do not introduce ghosts.

Following closely sections II and III, in section IV we show that in the spin-2 case, a similar change of variables can be made in the massive Fierz-Pauli theory formulated in terms of a non-symmetric rank-2 tensor. We deduce the D -dimensional linearized NMG model, see (49), and prove that it correctly describes a massive “spin-2” particle in arbitrary dimensions. We explain why the equations of motion of (49) can be written as a set of zero “curvature” conditions. In section V we recover the linearized NMG in $D = 3$ [13] and in $D = 4$ [16]. In section VI we draw our conclusions and perspectives.

2 Higher derivative spin-0 model

We can rewrite the Klein-Gordon action as a first-order theory by lowering the order of the massless kinetic term via a vector field:

$$S[\phi, A, J] = \int d^D x \left(\frac{m^2}{2} A_\mu A^\mu + m A_\mu \partial^\mu \phi - \frac{m^2}{2} \phi^2 + J^\mu A_\mu \right) . \quad (1)$$

We have introduced an arbitrary source term for the vector field for future purposes. Throughout this work we use $\eta_{\mu\nu} = \text{diag}(-, +, \dots, +)$.

If we integrate over the vector field in the corresponding functional generating function we obtain

$$S[\phi, J] = \int d^D x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{J^\mu \partial_\mu \phi}{m} - \frac{J_\mu J^\mu}{2m^2} \right) . \quad (2)$$

On the other hand, if we had first integrated over the scalar field we would have obtained a higher-rank description of a spin-0 particle,

$$S[A, J] = \int d^D x \left[\frac{1}{2} (\partial_\mu A^\mu)^2 + \frac{m^2}{2} A_\mu A^\mu + J^\mu A_\mu \right] . \quad (3)$$

It can be shown that (3) correctly describes a spin-0 massive particle. However, the massless limit of the vector theory does not describe a massless spin-0 particle as opposed to the massless limit of (2). The massless theory $\mathcal{L}_{m=0} \equiv (\partial_\mu A^\mu)^2 / 2$ has no particle content. It is invariant under gauge transformations

$$\delta_\Lambda A_\mu = \partial^\nu \Lambda_{\mu\nu} \quad ; \quad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} . \quad (4)$$

The equations of motion $\partial_\mu (\partial \cdot A) = 0$ lead, with vanishing fields at infinity, to $\partial \cdot A = 0$, so we are left with pure gauge degrees of freedom. It is convenient for our purposes to check the trivial content of $\mathcal{L}_{m=0}$ by lowering its order, see (1), , i.e., $\mathcal{L}_{m=0} = -\phi^2/2 - \phi \partial \cdot A$. The integral over the vector field leads to the non-dynamic effective Lagrangian $-\frac{\phi^2}{2}$ altogether with the functional delta function $\delta(\partial_\mu \phi)$ confirming the empty spectrum of $\mathcal{L}_{m=0}$.

Since the massless term of (3) has no particle content, it can be used as a “mixing term” in the master action approach of [17] in order to derive another physically equivalent action for a spin-0 particle. By introducing a dual vector field B_μ we have:

$$S_M[A, B, J] = \int d^D x \left\{ \frac{1}{2} (\partial_\mu A^\mu)^2 + \frac{m^2}{2} A_\mu A^\mu - \frac{1}{2} [\partial_\mu (B^\mu - A^\mu)]^2 + J^\mu A_\mu \right\} . \quad (5)$$

The shift $B_\mu \rightarrow B_\mu + A_\mu$ decouples the vector fields and since the kinetic term for B_μ has no propagating degree of freedom, it is clear that the particle content of (5) is the same of (3). On the other hand, if we integrate over A_μ we have a fourth-order description of a spin-0 particle:

$$S[B, J] = \int d^D x \left[\frac{1}{2m^2} \partial \cdot B (\square - m^2) \partial \cdot B + \frac{J^\mu \partial_\mu \partial \cdot B}{m^2} - \frac{J^\mu J_\mu}{2m^2} \right] . \quad (6)$$

The action $S[B]$ is invariant under the gauge transformation $\delta_\Lambda B_\mu$ given in (4). Assuming vanishing fields at infinity, the equations of motion of (6), at vanishing sources, imply

$$(\square - m^2) \partial \cdot B = 0 \quad . \quad (7)$$

which reproduces the Klein-Gordon equation for the gauge invariant scalar $\partial \cdot B$.

Regarding unitarity of (6) we now look at the two-point amplitude saturated with sources. The Λ -gauge symmetry allows us to fix the gauge $F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu = 0$. So we can add a gauge fixing term $-\lambda F_{\mu\nu}^2(B)$ to (6) and obtain the propagator. After the trivial redefinition $B_\mu \rightarrow \sqrt{2} m B_\mu$, the saturated two point function becomes

$$A(k) = \tilde{J}_\mu^*(k) \langle A^\mu(-k) A^\nu(k) \rangle \tilde{J}_\nu(k) = \frac{i}{2} \tilde{J}_\mu^* \left[\frac{\omega^{\mu\nu}}{k^2(k^2 + m^2)} + \frac{\theta^{\mu\nu}}{\lambda k^2} \right] \tilde{J}_\nu \quad . \quad (8)$$

where \tilde{J}_μ is the source for the B -field and the spin-0 and spin-1 projection operators are given respectively by

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \quad ; \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu} \quad . \quad (9)$$

The gauge invariance of the source term $\delta_\Lambda \int d^D x \tilde{J}^\mu B_\mu = 0$ requires longitudinal currents $\tilde{J}_\mu(k) = i k_\mu J(k)$. Back in (8) we have the gauge invariant amplitude for the B -model (6):

$$A(k) = \frac{i}{2} \frac{|J|^2}{(k^2 + m^2)} \quad . \quad (10)$$

The imaginary part of the residue at $k^2 = -m^2$ is positive $R_{-m^2} = |J|^2/2 > 0$. Therefore, we have one massive spin-0 physical particle in the spectrum in agreement with the original Klein-Gordon theory. Remarkably, after the redefinition $B_\mu \rightarrow \sqrt{2} m B_\mu$, the massless limit does also agree with the Klein-Gordon field theory. The 4th-order B -model corresponds to the original Klein-Gordon model (2) with the replacement $\phi \rightarrow -(\partial \cdot B)/m$. In order to understand that point it is instructive to define a triple master action

$$S_M[A, B, \phi] = \int d^D x \left\{ m A_\mu \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{m^2}{2} A_\mu A^\mu - \frac{1}{2} [\partial_\mu (B^\mu - A^\mu)]^2 + J^\mu A_\mu \right\} \quad . \quad (11)$$

If we first shift $B_\mu \rightarrow \tilde{B}_\mu + A_\mu$ and then integrate over \tilde{B}_μ and A_μ we obtain the effective action (2), whereas integrating over ϕ followed by the integral over A_μ gives rise to (6). From derivatives with respect to the source J^μ for the intermediate A_μ field we prove the equivalence of correlation functions

$$\langle \partial_{\mu_1} \phi(x_1) \cdots \partial_{\mu_N} \phi(x_N) \rangle_{S[\phi, 0]} = \frac{1}{(-m)^N} \langle \partial_{\mu_1} \partial \cdot B(x_1) \cdots \partial_{\mu_N} \partial \cdot B(x_N) \rangle_{S[B, 0]} \quad (12)$$

Since the quadratic terms in the sources in (2) and (6) are the same ones, the contact terms in (12) have canceled out. So the correlation functions tend to match even at coinciding points. So the dual map $\partial_\mu \phi(x) \leftrightarrow -\partial_\mu [\partial \cdot B(x)]/m$ holds strongly and can be substituted inside the action where we have coinciding points like $\partial_\mu \phi(x) \partial^\mu \phi(x)$. The previous map and the boundary condition of vanishing fields at infinity imply the map $\phi \leftrightarrow -\partial \cdot B/m$.

This is to be compared to other dual maps between free theories, like for instance, the map $f_\mu \leftrightarrow F_\mu \equiv \epsilon_{\mu\nu\beta} \partial^\nu A^\beta / m$ between the self-dual model of [18] and the Maxwell-Chern-Simons theory of [19]. The correlation functions of f_μ only coincide with correlation functions of F_μ up to contact terms, see [20]. If we replace $f_\mu = F_\mu$ directly in the self-dual model we end up with a third-order theory with ghosts in disagreement with Maxwell-Chern-Simons model of [19].

3 Higher derivative spin-1 model

Analogously to the last section, we first rewrite the Maxwell-Proca theory in a first-order formalism by lowering the order of the Maxwell theory with help of a symmetric tensor $W_{\mu\nu} = W_{\nu\mu}$ following the appendix of [21],

$$S[A, W, T] = \int d^D x \left[\frac{m^2}{2} \left(W_{\mu\nu} W^{\mu\nu} - \frac{W^2}{D-1} \right) + \sqrt{2} m W^{\mu\nu} \partial_\mu A_\nu - \frac{m^2}{2} A_\mu A^\mu + W_{\mu\nu} T^{\mu\nu} \right] . \quad (13)$$

where $T^{\mu\nu} = T^{\nu\mu}$ is the source.

Although the interacting term with the W -field depends only on the symmetric combination $\partial_{(\mu} A_{\nu)} = (\partial_\mu A_\nu + \partial_\nu A_\mu)/2$, if we integrate in the path integral over $W_{\mu\nu}$ we get the Maxwell-Proca theory

$$S_{MP}[A, T] = \int d^D x \left[-\frac{1}{4} F_{\mu\nu}^2 - \frac{m^2}{2} A_\mu A^\mu + \frac{\sqrt{2}}{m} T^{\mu\nu} (\eta_{\mu\nu} \partial \cdot A - \partial_{(\mu} A_{\nu)}) + \frac{T^2 - T_{\mu\nu}^2}{2 m^2} \right] . \quad (14)$$

where $T = \eta_{\mu\nu} T^{\mu\nu}$.

On the other hand, starting with (13) and integrating over the vector field A_μ we have:

$$S[W, T] = \int d^D x \left[(\partial^\mu W_{\mu\nu})^2 + \frac{m^2}{2} \left(W_{\mu\nu} W^{\mu\nu} - \frac{W^2}{D-1} \right) + W_{\mu\nu} T^{\mu\nu} \right] . \quad (15)$$

In [22] we have shown that (15) describes a massive spin-1 particle at classical and quantum level and that the massless limit, similar to (3), is singular. The Lagrangian $\mathcal{L}_{m=0} = (\partial^\mu W_{\mu\nu})^2$ has no particle content. The easiest way to check it is to lower its order rewriting it as $\mathcal{L}_{m=0} = -A_\mu A^\mu / 2 - A^\mu \partial^\nu W_{\mu\nu}$. Integrating over $W_{\mu\nu}$ we have $\partial_\mu A_\nu + \partial_\nu A_\mu = 0$ whose general solution, with vanishing fields at infinity, is trivial $A_\mu = 0$.

Based upon the similarities with the spin-0 case, the term $(\partial^\mu W_{\mu\nu})^2$ can be used as a “mixing term” in a triple master action with sources:

$$\begin{aligned} S[A, W, H, T] &= \int d^D x \left\{ \sqrt{2} m W^{\mu\nu} \partial_\mu A_\nu - \frac{m^2}{2} A_\mu A^\mu + \frac{m^2}{2} \left(W_{\mu\nu} W^{\mu\nu} - \frac{W^2}{D-1} \right) \right. \\ &\quad \left. - [\partial^\mu (H_{\mu\nu} - W_{\mu\nu})]^2 + W_{\mu\nu} T^{\mu\nu} \right\} . \end{aligned} \quad (16)$$

where we have introduced the symmetric dual field $H_{\mu\nu} = H_{\nu\mu}$. On one hand, if we shift $H_{\mu\nu} \rightarrow H_{\mu\nu} + W_{\mu\nu}$ and integrate over $H_{\mu\nu}$ and $W_{\mu\nu}$ we derive the Maxwell-Proca theory given

in (14), whereas the functional integration over A_μ followed by the integral over $W_{\mu\nu}$ leads to the 4th-order model:

$$S[H, T] = \int d^D x \left\{ -\frac{1}{4} F_{\mu\nu}^2 [\partial H] - \frac{m^2}{2} (\partial^\nu H_{\mu\nu})^2 - \frac{\sqrt{2} T^{\mu\nu}}{m} [\eta_{\mu\nu} \partial^\alpha \partial^\beta H_{\alpha\beta} - \partial^\alpha \partial_{(\mu} H_{\nu)\alpha}] + \frac{T^2 - T_{\mu\nu}^2}{2m^2} \right\} \quad (17)$$

where $F_{\mu\nu}[\partial H] = \partial_\mu(\partial^\alpha H_{\nu\alpha}) - \partial_\nu(\partial^\alpha H_{\mu\alpha})$.

As in the spin-0 case, the quadratic terms in the sources in (14) and (17) are exactly the same which leads us to identify correlation functions of $\eta_{\mu\nu} \partial \cdot A - \partial_{(\mu} A_{\nu)}$ in the Maxwell-Proca theory with correlation functions of $-\eta_{\mu\nu} \partial^\alpha \partial^\beta H_{\alpha\beta} + \partial^\alpha \partial_{(\mu} H_{\nu)\alpha}$ in the $S[H, 0]$ theory. The contact terms cancel out again. Consequently, we have the strong dual map $A_\mu \leftrightarrow -\partial^\alpha H_{\mu\alpha}$ which can be used inside Lagrangians. Thus, explaining the Maxwell-Proca form of (17). Notice also that $S[H, 0]$ is invariant under any local transformation which preserves $\partial^\alpha H_{\alpha\beta}$. They can be written [23] as $\delta_B H_{\mu\nu} = \partial^\sigma \partial^\rho B_{\mu\sigma\rho\nu}$ where the gauge parameters $B_{\mu\sigma\rho\nu}$ have the same index symmetries of the Rieman tensor.

The equations of motion of (17) are

$$\partial_\mu V_\nu + \partial_\nu V_\mu = 0 \quad ; \quad \text{where} \quad V_\nu \equiv [\eta_{\mu\nu}(\square - m^2) - \partial_\mu \partial_\nu] \partial_\sigma H^{\mu\sigma} \quad . \quad (18)$$

With vanishing fields at infinity we have the general solution $V_\mu = 0$ which is equivalent to the Maxwell-Proca equations with the identification of the vector field A_μ with the gauge invariant $\partial^\sigma H_{\mu\sigma}$, which proves the classical equivalence of the 4th-order model (17) with the Maxwell-Proca theory.

Concerning unitarity, first it is convenient to introduce sources for $H_{\mu\nu}$ and write (17) in terms of spin projection operators:

$$S[H, T] = \int d^D x \left\{ H_{\lambda\mu} \left[-\frac{\square(\square - m^2)}{2} P_{SS}^{(1)} + m^2 \square P_{WW}^{(0)} \right]_{\alpha\beta}^{\lambda\mu} H^{\alpha\beta} + H_{\mu\nu} \tilde{T}^{\mu\nu} \right\} \quad (19)$$

where the above spin-1 and spin-0 projection operators are given respectively by

$$\left(P_{SS}^{(1)} \right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{2} (\theta^\lambda_\alpha \omega^\mu_\beta + \theta^\mu_\alpha \omega^\lambda_\beta + \theta^\lambda_\beta \omega^\mu_\alpha + \theta^\mu_\beta \omega^\lambda_\alpha) \quad , \quad (20)$$

$$\left(P_{WW}^{(0)} \right)_{\alpha\beta}^{\lambda\mu} = \omega^{\lambda\mu} \omega_{\alpha\beta} \quad , \quad (21)$$

Secondly, we can add a gauge fixing term $\lambda G_{\mu\nu}^2(H)$ to the action, where

$$G_{\mu\nu}(H) \equiv \square H_{\mu\nu} - 2\partial^\alpha \partial_{(\mu} H_{\nu)\alpha} + \eta_{\mu\nu} \partial^\alpha \partial^\beta H_{\alpha\beta} = 0 \quad , \quad (22)$$

defines a good gauge condition. It is symmetric and transverse $\partial^\mu G_{\mu\nu} = 0$ just like the gauge parameter $\partial^\sigma \partial^\rho B_{\mu\sigma\rho\nu}$. Now we are able to obtain the propagator and the two-point amplitude saturated with sources. Since the gauge symmetry is such that $\delta_B \partial^\nu H_{\mu\nu} = 0$, the invariance of the source term $\delta_B \int d^D x H_{\mu\nu} \tilde{T}^{\mu\nu} = 0$ requires $\tilde{T}_{\mu\nu} = \partial_\mu J_\nu + \partial_\nu J_\mu$. Taking into account such restriction we have

$$\begin{aligned}
A(k) &= \tilde{T}_{\nu\alpha}^* \langle H^{\nu\alpha}(-k) H^{\sigma\mu}(k) \rangle \tilde{T}_{\mu\sigma}(k) = i \tilde{T}_{\nu\alpha}^* \left[\frac{2P_{SS}^{(1)}}{k^2(k^2 + m^2)} + \frac{P_{WW}^{(0)}}{m^2 k^2} \right]^{\nu\alpha\mu\sigma} \tilde{T}_{\mu\sigma} \\
&= i \left(\frac{J_\mu^*(k) \theta^{\mu\nu} J_\nu(k)}{k^2 + m^2} + \frac{J_\mu^*(k) \omega^{\mu\nu} J_\nu}{m^2} \right)
\end{aligned} \tag{23}$$

where we have used $\tilde{T}_{\mu\nu} = i(k_\mu J_\nu + k_\nu J_\mu)$. The last line of (23) corresponds exactly to the two-point amplitude of the Maxwell-Proca theory with sources J_μ . The pole at $k^2 = 0$ inside the operators (9) cancel out and the residue at $k^2 = -m^2$ is of course positive, see for instance [22], which guarantees unitarity of our higher derivative Maxwell-Proca theory.

Before finishing this section we point out that we could have started from a simpler first-order version of the Maxwell-Proca theory where an antisymmetric field $B_{\mu\nu} = -B_{\nu\mu}$ (2-form) is introduced to lower the order Maxwell theory instead of the symmetric field $W_{\mu\nu}$ of [21] and followed the same steps to arrive at a 4th-order theory. However, the 4th-order model would not be ghost-free. The point is that the kinetic term $(\partial_{[\mu} B_{\nu\rho]})^2$ does not have an empty spectrum. Thus, it can not be used as a mixing term in the master action approach. In fact, it is known that massless 2-forms are equivalent to massless $D - 4$ forms. In $D = 3$ we could have used a vector field to lower the order of the Maxwell theory but the dual model would have also a Maxwell Lagrangian as kinetic term which is equivalent on its turn to a massless scalar particle in $D = 3$, so invalidating our master action approach which crucially depends on the specific first-order formulation of the Maxwell theory introduced in [21]. In the next section we analyze the spin-2 case.

4 Linerarized “New Massive Gravity” via Ω -field

Although the minimal tensor structure to describe massive spin-2 particles is a symmetric rank-2 tensor, in order to derive a higher derivative unitary model it is convenient, see [16], to start with the first-order description of the Einstein-Hilbert theory (massless action) in terms of the vielbein and spin-connection. After linearization around a flat background and addition of the Fierz-Pauli mass term we have:

$$\begin{aligned}
S[\omega, e, J] &= \int d^D x [\omega^{\mu\nu\rho} \omega_{\rho\nu\mu} - \omega^\mu \omega_\mu + 2\omega_{\mu\nu\alpha} K^{\alpha\mu\nu}(e) + 2\omega_\mu K^\mu(e) \\
&\quad - m^2 (e_{\mu\nu} e^{\nu\mu} - e^2) + \omega_{\mu\nu\alpha} J^{\mu\nu\alpha}] .
\end{aligned} \tag{24}$$

where $e = e^\mu_\mu$ and $e_{\mu\nu}$ is a nonsymmetric tensor which might be understood as the fluctuation of the vielbein about a flat background while

$$\omega_{\mu\nu\alpha} = -\omega_{\mu\alpha\nu} \quad ; \quad \omega_\alpha = \eta^{\mu\nu} \omega_{\mu\nu\alpha} \quad , \tag{25}$$

$$K_{\alpha\mu\nu}(e) = -K_{\alpha\nu\mu}(e) = \partial_{[\alpha} e_{\mu]\nu} - \partial_{[\alpha} e_{\nu]\mu} + \partial_{[\nu} e_{\mu]\alpha} = \partial_\alpha e_{[\mu\nu]} - \partial_\mu e_{(\nu\alpha)} + \partial_\nu e_{\mu\alpha} \quad , \tag{26}$$

$$K_\nu(e) = \eta^{\alpha\mu} K_{\alpha\mu\nu}(e) = \partial_\nu e - \partial^\gamma e_{\nu\gamma} \quad . \tag{27}$$

The tensor $K_{\alpha\mu\nu}$ and consequently the action $S[\omega, e, 0]$ is invariant under linearized reparametrizations $\delta e_{\mu\nu} = \partial_\mu \xi_\nu$. There is also a local symmetry in (24) due to antisymmetric shifts $\delta e_{\mu\nu} = \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$, $\delta \omega_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho}$.

If we integrate over the “spin-connection” $\omega_{\mu\nu\rho}$ in the path integral we obtain the massive Fierz-Pauli theory \mathcal{L}_{FP} :

$$\begin{aligned} S_{FP}[e, J] &= \int d^D x [\mathcal{L}_{LEH}(e) - m^2 (e_{\mu\nu} e^{\nu\mu} - e^2) + K_{\mu\nu\alpha}(e) J^{\mu\nu\alpha} + \mathcal{L}_{JJ}] \\ &= \int d^D x [\mathcal{L}_{FP}(e) + K_{\mu\nu\alpha}(e) J^{\mu\nu\alpha} + \mathcal{L}_{JJ}] \quad , \end{aligned} \quad (28)$$

where the linearized Einstein-Hilbert theory and the quadratic terms in the sources are given by

$$\mathcal{L}_{LEH}(e) = K_\mu(e) K^\mu(e) + K_{\mu\nu\alpha}(e) K^{\alpha\mu\nu}(e) \quad , \quad (29)$$

$$\mathcal{L}_{JJ} = \frac{1}{2} J_{\mu\nu\alpha} J^{\nu\alpha\mu} - \frac{1}{4} J_{\mu\nu\alpha} J^{\mu\nu\alpha} + J_\mu J^\mu \quad . \quad (30)$$

On the other hand, if we start with (24) and integrate over $e_{\mu\nu}$ in the path integral we derive the dual model:

$$S[\omega, J] = \int d^D x [\mathcal{L}_{\omega\omega} + \omega^{\mu\nu\rho} w_{\rho\nu\mu} - \omega^\mu \omega_\mu + \omega_{\mu\nu\alpha} J^{\mu\nu\alpha}] \quad . \quad (31)$$

where

$$\mathcal{L}_{\omega\omega} = \frac{1}{m^2} \left[(\partial^\nu \omega_{\mu\alpha\nu}) - \frac{(\partial \cdot \omega)^2}{D-1} \right] \quad . \quad (32)$$

As in the previous two sections, the kinetic term $\mathcal{L}_{\omega\omega}$ has no particle content. This has been shown in [16] in the case $D = 4$ via a canonical analysis in a given gauge. Next we show that it can be generalized for arbitrary dimensions without fixing a gauge. Namely, we can rewrite $\mathcal{L}_{\omega\omega}$ in a first-order form

$$\mathcal{L}_{(1)} = 2\omega_{\mu\nu\rho} \partial^\nu e^{\rho\mu} + \omega_\mu (\partial^\mu e - \partial_\gamma e^{\mu\gamma}) - m^2 (e_{\mu\nu} e^{\nu\mu} - e^2) \quad , \quad (33)$$

$$= \mathcal{L}_{\omega\omega} - m^2 [(e_{\mu\nu} - E_{\mu\nu}) (e^{\nu\mu} - E^{\nu\mu}) - (e - E)^2] \quad , \quad (34)$$

where

$$E_{\mu\nu} = \frac{1}{m^2} \left(\partial_\mu \omega_\nu + \partial^\rho \omega_{\mu\nu\rho} - \frac{\eta_{\mu\nu}}{D-1} \partial \cdot \omega \right) \quad . \quad (35)$$

After the shift $e_{\mu\nu} \rightarrow e_{\mu\nu} + E_{\mu\nu}$, it becomes clear that $\mathcal{L}_{(1)}$ has the same particle content of $\mathcal{L}_{\omega\omega}$. On the other hand, integrating over $\omega_{\mu\nu\rho}$ in (33) we have a functional delta function assuring the constraint

$$\frac{\delta S_{(1)}}{\delta \omega_{\mu\nu\rho}} = \partial^\nu e^{\rho\mu} - \partial^\rho e^{\nu\mu} + \eta^{\mu\nu} (\partial^\rho e - \partial_\gamma e^{\rho\gamma}) - \eta^{\mu\rho} (\partial^\nu e - \partial_\gamma e^{\nu\gamma}) = 0 \quad . \quad (36)$$

whose general solution is $e_{\mu\nu} = \partial_\mu \phi_\nu$ with arbitrary ϕ_μ . This can be seen from $\eta_{\mu\nu} \frac{\delta S_{(1)}}{\delta \omega_{\mu\nu\rho}} = 0$ back in the constraint (36) which leads to $\partial^\nu e^{\rho\mu} - \partial^\rho e^{\nu\mu} = 0$. Since $\int d^D x (e_{\mu\nu} e^{\nu\mu} - e^2)$ vanishes at $e_{\mu\nu} = \partial_\mu \phi_\nu$ we conclude that $\mathcal{L}_{(1)}$ and consequently \mathcal{L}_{ww} has no particle content in arbitrary dimensions and as such it can be used as a “mixing term” in a triple master action,

$$\begin{aligned} S_M [e, \omega, \tilde{\Omega}, J] &= \int d^D x [\omega^{\mu\nu\rho} \omega_{\rho\nu\mu} - \omega^\mu \omega_\mu + 2\omega_{\mu\nu\alpha} K^{\alpha\mu\nu}(e) + 2\omega_\mu K^\mu(e) \\ &\quad - m^2 (e_{\mu\nu} e^{\nu\mu} - e^2) - (\mathcal{L}_{\omega\omega})_{\omega \rightarrow \omega + \tilde{\Omega}} + \omega_{\mu\nu\alpha} J^{\mu\nu\alpha}] . \end{aligned} \quad (37)$$

On one hand, after the shift $\tilde{\Omega}_{\mu\nu\rho} \rightarrow \tilde{\Omega}_{\mu\nu\rho} - \omega_{\mu\nu\rho}$ we can integrate over $\tilde{\Omega}_{\mu\nu\rho}$ and $\omega_{\mu\nu\rho}$. Then, we arrive at the massive Fierz-Pauli theory (28). On the other hand, integrating over $e_{\mu\nu}$ followed by the integral over $\omega_{\mu\nu\rho}$ we obtain a 4th-order model dual to the massive Fierz-Pauli theory:

$$\begin{aligned} S_{FP} [\bar{f}(\tilde{\Omega}), J] &= \int d^D x [\mathcal{L}_{KK}(\bar{f}) - m^2 (\bar{f}_{\mu\nu} \bar{f}^{\nu\mu} - \bar{f}^2) + K_{\mu\nu\alpha}(\bar{f}) J^{\mu\nu\alpha} + \mathcal{L}_{JJ}] \\ &= \int d^D x [\mathcal{L}_{FP}(\bar{f}) + K_{\mu\nu\alpha}(\bar{f}) J^{\mu\nu\alpha} + \mathcal{L}_{JJ}] , \end{aligned} \quad (38)$$

Where

$$\bar{f}_{\mu\nu} = -\frac{1}{m^2} \left(\partial^\rho \tilde{\Omega}_{\mu\nu\rho} - \frac{\eta_{\mu\nu}}{D-1} \partial \cdot \tilde{\Omega} \right) . \quad (39)$$

We can further simplify the dual model (38). Taking derivatives of the triple master action (37) with respect to the source $J^{\mu\nu\alpha}$ we derive the strong (without contact terms) equivalence between correlation functions:

$$\langle K_{\mu_1\nu_1\rho_1}[e(x_1)] \cdots K_{\mu_N\nu_N\rho_N}[e(x_N)] \rangle_{S_{FP}(e)} = \langle K_{\mu_1\nu_1\rho_1}[\bar{f}(x_1)] \cdots K_{\mu_N\nu_N\rho_N}[\bar{f}(x_N)] \rangle_{S_{FP}[\bar{f}(\tilde{\Omega})]} . \quad (40)$$

We infer the local dual map $K_{\mu\nu\rho}[e(x)] = K_{\mu\nu\rho}[\bar{f}(x)]$ which is equivalent to $K_{\mu\nu\rho}[e - \bar{f}] = 0$ whose general solution is pure gauge $(e - \bar{f})_{\mu\nu} = \partial_\mu \xi_\nu$. So we can write down

$$e_{\mu\nu} = \bar{f}_{\mu\nu} + \partial_\mu \xi_\nu = -\frac{1}{m^2} \left(\partial^\rho \tilde{\Omega}_{\mu\nu\rho} - \frac{\eta_{\mu\nu}}{D-1} \partial \cdot \tilde{\Omega} \right) + \partial_\mu \xi_\nu = \partial^\rho \Omega_{\mu\nu\rho} + \partial_\mu \tilde{\xi}_\nu , \quad (41)$$

where

$$\Omega_{\mu\nu\rho} = -\frac{1}{m^2} \left(\tilde{\Omega}_{\mu\nu\rho} + \frac{\eta_{\mu\rho} \tilde{\Omega}_\nu - \eta_{\mu\nu} \tilde{\Omega}_\rho}{D-1} \right) , \quad (42)$$

$$\tilde{\xi} = \xi_\nu + \frac{\tilde{\Omega}_\nu}{m^2(D-1)} . \quad (43)$$

Thus, we deduce the local dual map between the nonsymmetric tensor $e_{\mu\nu}$ of the Fierz-Pauli theory and the traceless ($\eta^{\mu\nu} \Omega_{\mu\nu\rho} = \Omega_\rho = 0$) field $\Omega_{\mu\nu\rho}$:

$$e_{\mu\nu} = f_{\mu\nu} + \partial_\mu \tilde{\xi}_\nu \quad , \quad (44)$$

with

$$f_{\mu\nu} = \partial^\rho \Omega_{\mu\nu\rho} \quad , \quad (45)$$

$$\eta^{\mu\nu} f_{\mu\nu} = f = 0 \quad , \quad (46)$$

$$\partial^\nu f_{\mu\nu} = 0 \quad . \quad (47)$$

Due to the subsidiary conditions (46) and (47), when we substitute $e_{\mu\nu} = f_{\mu\nu} + \partial_\mu \tilde{\xi}_\nu$ in the massive Fierz-Pauli theory (28), the arbitrary functions $\tilde{\xi}_\nu$ drop out and we recover the 4th-order dual model (38) with $\bar{f}_{\mu\nu}$ replaced by $f_{\mu\nu}$. Dropping the source terms, the linearized “New Massive Gravity” Lagrangian in arbitrary dimensions can be written as¹

$$\mathcal{L}_\Omega = \frac{1}{2} (\partial^\mu f_{\mu\nu}) (\partial^\alpha f_\alpha{}^\nu) - \partial^\nu f^{(\alpha\beta)} \partial_\nu f_{(\alpha\beta)} - m^2 f_{\mu\nu} f^{\nu\mu} \quad , \quad (48)$$

$$= \frac{1}{2} (\partial^\mu \partial^\rho \Omega_{\mu\nu\rho})^2 - [\partial^\nu \partial^\rho \Omega_{(\alpha\beta)\rho}]^2 - m^2 (\partial^\rho \Omega_{\mu\nu\rho}) (\partial^\alpha \Omega^{\nu\mu}{}_\alpha) \quad . \quad (49)$$

Although our triple master action approach already proves that \mathcal{L}_Ω only contains one massive “spin-2” particle in the spectrum, it is instructive to check it directly from \mathcal{L}_Ω as follows.

First, we note that the equations of motion of \mathcal{L}_Ω can be compactly written as a set of zero “curvature” conditions:

$$\partial_\rho T_{\beta\alpha} - \partial_\beta T_{\rho\alpha} = 0 \quad . \quad (50)$$

where

$$T_{\mu\nu} = \square f_{(\mu\nu)} - m^2 f_{\mu\nu} - \frac{1}{2} \partial_\nu \partial^\beta f_{\beta\mu} \quad , \quad (51)$$

$$T = 0 = \partial^\nu T_{\mu\nu} \quad , \quad (52)$$

with $f_{\mu\nu}$ given in (45). The simplicity of (50) is connected with the fact that $S_\Omega = \int d^D x \mathcal{L}_\Omega$ depends upon the traceless tensor $\Omega_{\mu\nu\rho}$ only through $f_{\mu\nu} = \partial^\rho \Omega_{\mu\nu\rho}$. One can deduce (50) either directly from (49) or via the functional chain rule:

$$\frac{\delta S_\Omega}{\delta \Omega_{\mu\nu\rho}(x)} = \int d^D z \frac{\delta S_\Omega}{\delta f_{\alpha\beta}(z)} \frac{\delta f_{\alpha\beta}(z)}{\delta \Omega_{\mu\nu\rho}(x)} = -\frac{1}{2} \left(\partial^\rho \frac{\delta S_\Omega}{\delta f_{\mu\nu}(x)} - \partial^\nu \frac{\delta S_\Omega}{\delta f_{\mu\rho}(x)} \right) \quad . \quad (53)$$

In order to prove (53) we have used the properties $\Omega_\rho = 0$, $\Omega_{\mu\nu\rho} = -\Omega_{\mu\rho\nu}$ and the identities $\eta^{\mu\nu} (\delta S_\Omega / \delta f_{\mu\nu}) = 0$, $\partial_\mu (\delta S_\Omega / \delta f_{\mu\nu}) = 0$.

¹Since the first two terms of (49) only depend upon $\Omega_{(\mu\nu)\rho}$, it is tempting to split $\Omega_{\mu\nu\rho} = \Omega_{[\mu\nu]\rho} + \Omega_{(\mu\nu)\rho}$ and drop $\Omega_{[\mu\nu]\rho}$. However, the corresponding theory in terms of $\Omega_{(\mu\nu)\rho}$ contains ghosts. The point is that $\Omega_{[\mu\nu]\rho}$ and $\Omega_{(\mu\nu)\rho}$ are not independent variables due to the property $\Omega_{\mu\nu\rho} = -\Omega_{\mu\rho\nu}$. For instance, $\Omega_{(13)1} = \Omega_{[13]1}$.

The general solution to (50) consistent with (52) is given in terms of an arbitrary transverse vector,

$$T_{\mu\nu} = \square f_{(\mu\nu)} - m^2 f_{\mu\nu} - \frac{1}{2} \partial_\nu \partial^\beta f_{\beta\mu} = \partial_\mu A_\nu^T \quad , \quad \partial^\mu A_\mu^T = 0 \quad , \quad (54)$$

In order to solve (54) we first notice that S_Ω is invariant under the gauge transformations:

$$\delta \Omega_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \frac{\eta_{\mu\rho} \partial^\alpha B_{\alpha\nu} - \eta_{\mu\nu} \partial^\alpha B_{\alpha\rho}}{D-1} + \partial^\alpha \Lambda_{[\alpha\nu\rho]\mu} \quad , \quad (55)$$

where $B_{\mu\nu} = -B_{\nu\mu}$ and $\partial^\alpha \Lambda_{[\alpha\mu\rho]}{}^\mu = 0$. After a trivial redefinition of $B_{\mu\nu}$ we have $\delta f_{\mu\nu} = \partial_\mu (\partial^\alpha B_{\alpha\nu})$. Consistency with (54) requires :

$$\delta A_\mu^T = \left(\frac{\square}{2} - m^2 \right) \partial^\nu B_{\nu\mu} \quad . \quad (56)$$

In order to find a convenient gauge condition we notice from $\delta f_{\mu\alpha} = \partial_\mu (\partial^\nu B_{\nu\alpha})$ and from ∂^μ applied on (54), respectively, that

$$\delta \partial^\mu f_{\mu\alpha} = \square (\partial^\nu B_{\nu\alpha}) \equiv \square U_\alpha^T \quad ; \quad \partial^\mu f_{\mu\alpha} = \frac{\square}{m^2} \left(\frac{\partial^\mu f_{\mu\alpha}}{2} - A_\alpha^T \right) \equiv \square V_\alpha^T \quad , \quad (57)$$

From (57) it is natural to fix the gauge:

$$\partial^\mu f_{\mu\alpha} = 0 \quad . \quad (58)$$

Back in the second equation of (57) we have $\square A_\mu^T = 0$. All the equations so far are invariant under residual harmonic gauge transformations: $\square (\partial^\mu B_{\mu\alpha}) = 0$. From (56) we have $\delta A_\alpha^T = -m^2 (\partial^\mu B_{\mu\alpha}) = -m^2 U_\alpha^T$. So we can use the residual symmetry to set $A_\alpha^T = 0$. The first equation (54) becomes $\square f_{(\nu\mu)} - m^2 f_{\mu\nu} = 0$ which leads to

$$f_{[\mu\nu]} = 0 \quad , \quad (\square - m^2) f_{\mu\nu} = 0 \quad . \quad (59)$$

The equations (59) and the identities $f = 0 = \partial^\mu f_{\alpha\mu}$ correspond to the Fierz-Pauli conditions. Since $f_{\mu\nu} = \partial^\rho \Omega_{\mu\nu\rho}$ is invariant under the unfixed gauge transformations $\delta \Omega_{\mu\nu\rho} = \partial^\alpha \Lambda_{[\alpha\nu\rho]\mu}$ where $\partial^\alpha \Lambda_{[\alpha\mu\rho]}{}^\mu = 0$, the particle content of S_Ω corresponds indeed to one massive “spin-2” particle as expected.

5 Recovering “New Massive” 3D and 4D Gravity

In order to make contact with the known theories of “New Massive Gravity” in $D = 2 + 1$ [13] and in $D = 3 + 1$ [16] we remark that the traceless Ω -field has $D(D+1)(D-2)/2$ independent components. In $D = 3$ we have 6 degrees of freedom which coincides with a symmetric rank-2 tensor. Inspired by [16] we use the Levi-Civita symbol and define the linear change of variables

$$\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho\alpha} h^\alpha{}_\mu \quad \rightarrow \quad f_{\mu\nu} = \partial^\rho \Omega_{\mu\nu\rho} = -\hat{E}_\nu{}^\alpha h_{\alpha\mu} \quad (60)$$

where $\hat{E}_{\nu\alpha} = \epsilon_{\nu\alpha\gamma} \partial^\gamma$. Due to the traceless condition $\eta^{\mu\nu} \Omega_{\mu\nu\rho} = 0$ we must have $h_{\mu\nu} = h_{\nu\mu}$. By replacing $f_{\mu\nu} = -\hat{E}_\nu{}^\alpha h_{\alpha\mu}$ in (49) and using the identity $\hat{E}_{\mu\alpha} \hat{E}_{\beta\nu} = \square (\theta_{\mu\nu} \theta_{\alpha\beta} - \theta_{\mu\beta} \theta_{\alpha\nu})$ we

arrive (up to an overall factor $2m^2$) at the linearized version of the new massive gravity of [13]:

$$\mathcal{L}_{NMG}^{3D} = \frac{1}{2} h_{\mu\beta} \square^2 [2\theta^{\mu\nu} \theta^{\alpha\beta} - \theta^{\mu\beta} \theta^{\alpha\nu}] h_{\nu\alpha} - m^2 h_{\mu\beta} \hat{E}^{\mu\alpha} \hat{E}^{\beta\nu} h_{\alpha\nu}. \quad (61)$$

The linearized reparametrization invariance $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ of S_{NMG}^{3D} becomes $\delta f_{\mu\nu} = \partial_\mu \Phi_\nu^T$ where $\Phi_\nu^T = -\hat{E}_{\nu\beta} \xi^\beta$ which is of course a symmetry of (49) as a consequence of the $B_{\mu\nu}$ gauge invariance (55). Notice also that in $D = 2 + 1$ the Λ -symmetry, see (55), is a subset of the $B_{\mu\nu}$ -symmetry.

In $D = 4$ the analogue of the map (60) is given by

$$\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho}{}^{\gamma\delta} T_{[\gamma\delta]\mu} \rightarrow f_{\mu\nu} = \partial^\rho \Omega_{\mu\nu\rho} = \hat{E}_\nu{}^{\gamma\delta} T_{[\gamma\delta]\mu} \quad (62)$$

where $\hat{E}_{\mu\nu\alpha} = \epsilon_{\mu\nu\alpha\gamma} \partial^\gamma$. The traceless condition $\Omega_\mu = 0$ requires $T_{[[\gamma\delta]\mu]} = 0$. The Ω -field has 20 components in $D = 4$ which is the same number of components of a tensor $T_{[\gamma\delta]\mu}$ without totally antisymmetric part. If we replace $\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho}{}^{\gamma\delta} T_{[\gamma\delta]\mu}$ in (49) we have

$$\mathcal{L} = T_{[\mu\nu]\beta} \left(\frac{\square}{2} - m^2 \right) G^{[\mu\nu]\beta}(T) - \frac{1}{4} T_{[\mu\nu]\alpha} \square \theta^{\alpha\delta} \hat{E}^{\mu\nu}{}_\beta E^{\lambda\epsilon\beta} T_{[\lambda\epsilon]\delta}. \quad (63)$$

where, see [16], $G^{[\mu\nu]\rho}(T) = -\hat{E}^{[\mu\nu]\beta} \hat{E}^{[\rho\delta]\epsilon} T_{[\delta\epsilon]\beta} / 2$. In order to bring (63) to the form given in [16] we have to rearrange its last term. We have found convenient to use the 4D identity $\epsilon_{[\mu\nu\beta\gamma]a_\delta} = 0$ twice. First with $a_\delta \rightarrow \partial_\delta$ and secondly with $a_\delta \rightarrow \eta_{\delta\alpha}$ and multiplying the result with ∂^γ . Back in (63) we get the linearized “New Massive” 4D gravity (up to an overall factor $2m^2$) as given in formula (29) of [16]:

$$\mathcal{L}_{LNMG}^{4D} = T_{[\mu\nu]\beta} (\square - m^2) G^{[\mu\nu]\beta}(T) + T_{[\mu\nu]\beta} \square \theta^{\beta[\mu} G^{\nu]}(T) \quad , \quad (64)$$

where $G_\mu(T) = \eta^{\nu\rho} G_{[\mu\nu]\rho}(T)$. It can be shown that the local symmetries of (64) mentioned in [16] are equivalent to the gauge symmetries (55).

For arbitrary dimensions we can introduce a rank-(D-1) tensor and derive dual descriptions via $\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho}{}^{\alpha_1 \dots \alpha_{D-2}} T_{[\alpha_1 \dots \alpha_{D-2}]\mu}$. Due to the traceless condition we must have $T_{[\alpha_1 \dots \alpha_{D-2}]\mu} = 0$. However, for $D \geq 5$ we better stick, for simplicity, to the rank-3 description given by \mathcal{L}_Ω .

Last, we comment on the linearized Weyl symmetry of 4th-order terms of S_Ω which is specially relevant for a possible nonlinear completion of S_Ω and its renormalizability.

The linearized Weyl symmetry of the 4th-order terms of S_{NMG}^{3D} and S_{NMG}^{4D} correspond respectively to the transformations $\delta_w h_{\mu\nu} = \eta_{\mu\nu} \phi$ and $\delta_w T_{[\alpha\beta]\mu} = (\eta_{\alpha\mu} \phi_\beta - \eta_{\beta\mu} \phi_\alpha) / 2$. In terms of the Ω -field we have $\delta_w \Omega_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} \phi$ and $\delta_w \Omega_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\beta} \phi^\beta$. They can be easily generalized to arbitrary dimensions:

$$\delta_w \Omega_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\alpha_1 \dots \alpha_{D-3}} \phi^{\alpha_1 \dots \alpha_{D-3}} \rightarrow \delta_w f_{\mu\nu} = -(-)^D \hat{E}_{\mu\nu\alpha_1 \dots \alpha_{D-3}} \phi^{\alpha_1 \dots \alpha_{D-3}}. \quad (65)$$

where $\hat{E}_{\mu\nu\alpha_1 \dots \alpha_{D-3}} = \epsilon_{\mu\nu\alpha_1 \dots \alpha_{D-3}\gamma} \partial^\gamma$.

Since the 4th-order terms of (48) depend upon $\partial^\mu f_{\mu\nu}$ and $f_{(\mu\nu)}$ it is clear they are invariant under the Weyl transformations (65) which are broken by the 2nd-order mass term. Thus, in arbitrary dimensions, a possible nonlinear completion of the “New Massive Gravity” \mathcal{L}_Ω may

have renormalizability problems, see [24] for the 3D NMG case. Since part of the degrees of freedom of the Ω -field will not be present in the 4th-order terms, their UV behavior will be ruled by the 2nd-order mass term $\sim 1/p^2$ which is a problem for renormalizability unless the nonlinear (self-interacting) terms are also Weyl invariant. This is not the case of the NMG in $D = 3$. See comments in a similar vein in [25, 26].

6 Conclusion and Outlook

Here we have shown that a triple master action approach can be used to deduce a higher-rank 4th-order (in derivatives) description of spin-0, spin-1 and spin-2 massive particles in arbitrary dimensions. By adding sources we have explained why the dual higher-derivative theories can be obtained via a strong local map which contains one derivative of a higher rank field and works directly at action level. This is to be contrasted with other examples in the literature where local maps between theories of different number of derivatives only hold at the level of equations of motion like, e.g., the duality in $D = 3$ between the spin-1 self-dual model [18] and the Maxwell-Chern-Simons theory where the map $f_\mu = \epsilon_{\mu\nu\beta}\partial^\nu A^\beta/m$ works on shell but leads to ghosts if used at action level (off-shell). The key point for the off-shell usefulness of the dual map is the absence of contact terms in the equivalence between correlation functions.

In the spin-2 case we have shown that the replacement $e_{\mu\nu} = \partial^\rho\Omega_{\mu\nu\rho}$ in the massive Fierz-Pauli action with a nonsymmetric rank-2 tensor leads directly to the unitary 4th-order model \mathcal{L}_Ω , see (49), which might be interpreted as a generalization of the linearized “New Massive” 3D and 4D gravity to arbitrary dimensions. We have shown that \mathcal{L}_Ω indeed describes a massive “spin-2” particle in arbitrary dimensions and reduces to the known “New Massive” gravity theories in $D = 3$ and $D = 4$. We have pointed out the importance of the traceless condition $\eta^{\mu\nu}\Omega_{\mu\nu\rho} = 0$ in order to simplify the dual 4th-order model and reduce the number of initial degrees of freedom. The equations of motion can be written in a simple form as zero “curvature” conditions.

Since the structure of the theory in D -dimensions is basically the same of the $D = 3$ case, i.e., one second-order term plus a “Weyl” invariant 4th-order term, it is expected that a possible nonlinear completion of \mathcal{L}_Ω might suffer from the same renormalizability problems of the $D = 3$ case, see [24] and comments in [25, 26], worsened by the higher dimensionality.

Even if we are able to add Weyl invariant interactions to \mathcal{L}_Ω , this will probably require the vertices to depend upon $f_{(\mu\nu)} = \partial^\rho\Omega_{(\mu\nu)\rho}$ which contains already one derivative such that the net gain in the power counting is zero. The same happens already in the spin-0 case treated in the second section. If we plug $\phi = \partial \cdot B$ in nonlinear terms in the scalar field, the extra $1/p^2$ factor in the UV behavior of the vector field propagator will be canceled by the extra p^2 factor on each internal line of Feynman diagrams due to the derivative vertices. So we better look for alternative higher-derivative theories where all propagating degrees of freedom are present in the highest derivative term.

It may be useful to comment that if we make a weak field expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in the nonlinear “New Massive” gravity of [13] in $D = 3$ and replace $h_{\mu\nu} = h_{\mu\nu}(\Omega)$ where $h_{\mu\nu}(\Omega)$ is obtained by inverting the map $\Omega_{\mu\nu\rho} = \epsilon_{\nu\rho\beta}h^\beta_\mu$, we obtain a consistent ghost-free (beyond

tree level, see [14]) interacting theory for the mixed symmetry Ω -field. Such formulation may be eventually relevant for the addition of interactions to \mathcal{L}_Ω for $D > 3$. Another possible approach is to look for a nonlinear completion of the gauge transformations (55) altogether with consistent (gauge invariant) vertices to be added to \mathcal{L}_Ω .

A further direction to follow is the generalization of the triple master action approach to higher-spin theories.

Finally, while typing this work we became aware of [27] where the issue of a linearized D -dimensional “New Massive Gravity” has also been addressed.

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